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Generalized Eady waves

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Solutions are obtained for the baroclinic instability problem for situations in which the static stability and mean shear vary geminately with height. The simple solution given by Eady is shown to be a special limiting case of a class of exact solutions for flows whose basic states have a vanishing interior potential vorticity gradient. The generalized solutions show that the temperature amplitude distribution is particularly sensitive to vertical variations in static stability but that phases and other amplitudes are only slightly influenced by such variations. When the static stability and shear increase (decrease) with height an enhanced temperature maximum occurs at the upper (lower) surface in comparison with the standard Eady solution.

The generalized solutions also help to explain the character of annulus waves and predict a short-wave cut-off that is the same as that given by Eady's theory provided that it is based on the vertically averaged gravitational frequency.

1. Introduction

The theory of baroclinic instability has mainly been developed for idealized mean currents that have a constant shear and static stability. This elementary configuration makes it possible for the problem to be solved analytically. The simplest such solution is that given by Eady (1949) in terms of hyperbolic functions.

The purpose of this paper is to show that simple analytical solutions exist, also in the form of hyperbolic functions, when the static stability and shear are non-constant but have the same functional variation with height. Such distributions maintain the vanishing interior potential vorticity gradient and the mathematical simplicity of the original Eady problem. The solutions allow us to examine the effects of variations in static stability and shear on the character of this particular set of baroclinic waves.

This analysis was originally made to try to explain the baroclinic annulus wave discussed earlier by the author (Williams 1971). Although these annulus waves are very similar in many respects to the standard Eady wave, they do, however, display a quite different temperature amplitude distribution, having a maximum in the lower fluid and a minimum at the upper surface. Because annulus waves are so similar in many respects to the classical Eady wave, yet have a significant deviation, they suggest the existence of a wider class of solutions that have the Eady solution as a particular case. Such solutions were sought.

The generalized solutions support the hypothesis that the deviation of the annulus wave from the Eady wave is due to the vertical variation in the static stability. The solutions also have possible geophysical relevance, particularly for the ocean, a system in which the static stability undergoes large variation with height.

2. The generalized Eady problem

The formulation of the baroclinic instability problem and the derivation of the governing equations have been well documented; see, for example, McIntyre (1970) for an up-to-date discussion. The problem is outlined below to introduce the notation.

We consider small amplitude inviscid adiabatic perturbations to a parallel flow $u(y,z)$ in the x direction of a channel limited by boundaries at $z = 0$, H and $y = 0$, L on which the normal velocities must vanish. The Cartesian co-ordinates (x, y, z) are in a frame of reference rotating about the vertical z axis with angular velocity $\frac{1}{2}f$. We take $\mathscr{S}_\epsilon$ and $\mathscr{N}_\epsilon$ to be characteristic values of the shear u_z (a frequency) and of the gravitational (Brunt–Väisälä) frequency $N(z) = (\beta g\, dT_s/dz)^{\frac{1}{2}}$ for a stably stratified Boussinesq fluid. The subscript ϵ denotes, for reasons that will become apparent, the non-constancy of these frequencies with respect to z.

For this particular baroclinic instability problem the following scaling is most appropriate:

$\mathscr{N}_\epsilon H/f$	for	the horizontal co-ordinates (x, y), a scaling suggested by Stone's (1969) analysis,	(1)
H	for	the vertical co-ordinate z,	
$\mathscr{N}_\epsilon/f\mathscr{S}_\epsilon$	for	the time t,	
$\mathscr{S}_\epsilon H$	for	the horizontal velocities (u, v),	
fHR^2	for	the vertical velocity w,	
$H^2\mathscr{N}_\epsilon\mathscr{S}_\epsilon$	for	the Boussinesq pressure p/ρ,	
$H\mathscr{N}_\epsilon\mathscr{S}_\epsilon$	for	the Boussinesq buoyancy $\sigma = -\beta gT$.	

In the quasi-static quasi-geostrophic state the Rossby–Kibel number $R = \mathscr{S}_\epsilon/\mathscr{N}_\epsilon$ (the reciprocal of the square root of the Richardson number) is small and the governing equations are those for the conservation of quasi-geostrophic potential vorticity q and for the advection of buoyancy σ.

When these equations are linearized to describe normal-mode perturbations of the basic state they can be written on transposing the variables to non-dimensional form as

$$\hat{q} + [q_y \hat{\psi}/(u-c)] = 0, \tag{2}$$

$$\hat{q} = (F\hat{\psi}_z)_z + \hat{\psi}_{yy} - \alpha^2\hat{\psi}, \tag{3}$$

$$q_y = -(Fu_z)_z - u_{yy}, \tag{4}$$

$$\hat{w} = i\alpha F\,[u_z\hat{\psi} - (u-c)\,\hat{\psi}_z], \tag{5}$$

where $\hat{q}$, $\hat{\psi}$ and $\hat{w}$ are the complex amplitude functions for a perturbation of the form $\psi = \mathrm{Re}\,\hat{\psi}(y,z)\,e^{i\alpha(x-ct)}$. The non-dimensional wavenumber α is a real variable, whereas the non-dimensional phase velocity $c = c_r + ic_i$ can be complex. The stream function ψ for the dimensionless horizontal velocities is also the quasi-geostrophic pressure, whereas its gradient ψ_z, by virtue of the hydrostatic relation $\sigma = -\psi_z$, describes the quasi-geostrophic temperature. The parameter $F(z) = \mathcal{N}_\epsilon^2/N^2(z)$ is the basic baroclinic instability parameter, sometimes called the rotational Froude number.

For the problem in hand u is assumed to be independent of the lateral coordinate y so that q_y is independent of y and solutions of the form $\hat{\psi}(y,z) = \hat{\psi}(z) \times \sin my$ exist to satisfy the lateral boundary conditions. Then equation (2) for $\hat{\psi}(z)$ becomes, for any $F(z)$ and $u(z)$,

$$(F\hat{\psi}_z)_z - [(Fu_z)_z/(u-c) + (\alpha^2+m^2)]\,\hat{\psi} = 0. \tag{6}$$

The standard Eady approximation eliminates the singular term $(u-c)^{-1}$ by assuming that $F(z)$ and u_z are both constant. This, however, is a redundant idealization for it is only necessary to assume that the product Fu_z is constant in order to make $q_y = 0$ and eliminate the singular term. We shall make this latter assumption and refer to the subsequent problem for the sake of definition as a generalized Eady problem.

To satisfy the generalized constraint the z variation of the basic state is taken to be of the form

$$N(z) = \mathcal{N}_0 n_\epsilon(z), \quad u_z(z) = \mathcal{S}_0 s_\epsilon(z)/\mathcal{S}_\epsilon, \tag{7}$$

such that $n_\epsilon^2(z) = s_\epsilon(z)$.† Thus the ensuing analysis is valid only for flows in which the mean shear and the static stability have parallel distributions in z. The constants $\mathcal{N}_0$ and $\mathcal{S}_0$ are the characteristic frequencies for a uniform basic state, denoted as $\epsilon = 0$. It no longer suffices in the case of a z-dependent basic state to base the characteristic values on single representative values of the parameters. Instead it is necessary to introduce characteristic values based on integrals of the basic state. Hindsight indicates that the appropriate integral scaling factor is the mean value of $n_\epsilon(z)$, i.e.

$$\bar{n}_\epsilon = \int_0^1 n_\epsilon(z)\,dz.$$

Thus the characteristic values are taken to be such that

$$\mathcal{N}_\epsilon^2/\mathcal{N}_0^2 = \mathcal{S}_\epsilon/\mathcal{S}_0 = \bar{n}_\epsilon^2.‡$$

Equation (6) for the conservation of potential vorticity in the fluid interior can then be written as

$$[\hat{\psi}_z/n_\epsilon^2(z)]_z = [4k^2/\bar{n}_\epsilon^2]\,\hat{\psi}, \tag{8}$$

† Despite this equivalence, both functions are retained to facilitate physical identification.

‡ This introduces integral factors $\bar{n}_\epsilon, 1, \bar{n}_\epsilon^{-1}$, $\bar{n}_\epsilon^2$, $\bar{n}_\epsilon^2$ $\bar{n}_\epsilon^3$ and $\bar{n}_\epsilon^3$ into the scales of (1). Alternatively, some simplification could be realized by normalizing the function $n_\epsilon(z)$.

where $4k^2 = \alpha^2 + m^2$. The variation in the shear enters the problem through the boundary condition, $\hat{w} = 0$, i.e.

$$u_z \hat{\psi} - (u-c)\hat{\psi}_z = 0, \tag{9}$$

that describes the temperature advection on the horizontal boundaries $z = 0, 1$.

3. Approximate and exact solutions

To solve (8) a new independent variable η is introduced such that $\bar{n}_\epsilon^2 d\eta = n_\epsilon^2(z)\,dz$. Then (8) can be transformed into the normalized Sturm–Liouville equation

$$\hat{\psi}_{\eta\eta} = E(\eta)\,\hat{\psi}, \tag{10}$$

where $E(\eta) = 4k^2 \bar{n}_\epsilon^2 n_\epsilon^{-2}(z)$. This equation has solutions expressible in terms of the functions of classical physics for various idealized forms of $E(\eta)$. However, to find a class of simple exact solutions to this problem we use only the results of WKB theory and choose $E(\eta)$ profiles such that the WKB error term vanishes. For $E(\eta) > 0$ approximate solutions to (10) are given by the WKB method in the form

$$\hat{\psi} = E^{-\frac{1}{4}}\{A_1 \exp[\textstyle\int E^{\frac{1}{2}}\,d\eta] + A_2 \exp[-\int E^{\frac{1}{2}}\,d\eta]\}, \tag{11}$$

where there is an associated error given by the expression

$$\tfrac{1}{2}E^{-\frac{1}{4}}(E^{-\frac{1}{4}})_{\eta\eta}. \tag{12}$$

Although these equations would allow approximate solutions to be obtained for any $n_\epsilon(z)$ consistent with the WKB method we are mainly interested in using (11) and (12) to reveal the most general form of *simple exact* solutions of this type. The exact solutions provide a simpler illustration of the effects of the z variation of the basic state. The more complex eigenvalue problem for the arbitrary $n_\epsilon(z)$ case is discussed in the appendix.

To obtain the most general exact solutions, (12) is solved for the case of zero error. The only forms of $E(\eta)$ that satisfy this condition are the polynomials $(a_1 + a_2\eta)^{-4}$, where a_1 and a_2 are arbitrary constants. The associated solutions are

$$(a_1 + a_2\eta)\exp[\pm\eta/a_1(a_1 + a_2\eta)]. \tag{13}$$

Thus simple exact solutions for the generalized Eady problem exist when $n_\epsilon^2(z)$ takes the form $(a_3 + a_4 z)^{-\frac{4}{3}}$, where a_3 and a_4 are arbitrary constants. (This is obtained on transforming to the original variables.) Therefore we assume that $n_\epsilon^2(z) = (1-\epsilon z)^{-\frac{4}{3}}$ so as to obtain the most general exact hyperbolic solutions to the modified Eady problem. The standard Eady problem corresponds to the case $\epsilon = 0$. The significance of the subscript ϵ is now apparent.

4. Solution of the generalized exact Eady problem

The generalized Eady problem can be defined for the exact case as being the study of the stability of a mean flow which has a static-stability–shear functional variation $n_\epsilon^2(z) = s_\epsilon(z) = (1-\epsilon z)^{-\frac{4}{3}}$ and a mean current given by (upon integrating)

$$u(z) = \mathscr{S}_0 \mathscr{S}_\epsilon^{-\frac{1}{2}}[(1-\epsilon z)^{-\frac{1}{3}} - 1]/\tfrac{1}{3}\epsilon, \tag{14}$$

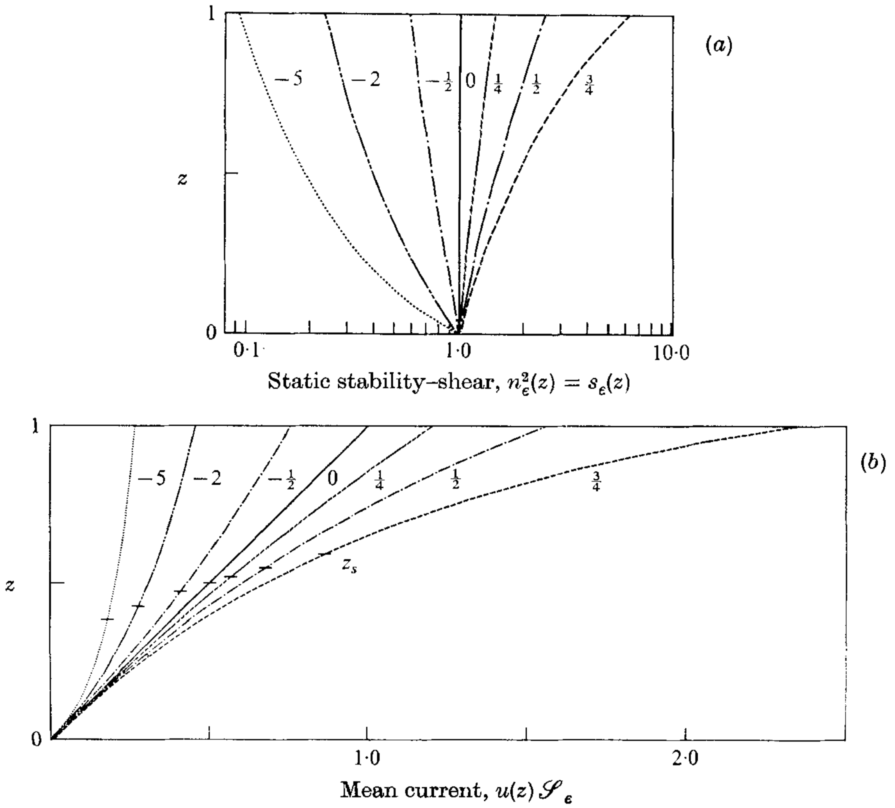

FIGURE 1. Distribution of (*a*) the static-stability–shear function $n_\epsilon^2(z) = s_\epsilon(z) = (1-\epsilon z)^{-\frac{4}{3}}$ and (*b*) the dimensional mean current $u(z)\,\mathscr{S}_\epsilon$, equation (14) with $\mathscr{S}_0 = 1$, with the intercepts indicating the steering level z_s, equation (26), for representative values of ϵ as indicated.

where $u(0) = 0$ is imposed. The forms of $n_\epsilon^2(z)$ and $u(z)$ are shown in figure 1 for cases of increasing ($\epsilon > 0$) and decreasing ($\epsilon < 0$) static stability and shear. The selected values of ϵ give static stabilities that increase or decrease by a factor of up to 10.† Clearly the functional forms represent simple but realistic distributions ($s_\epsilon(z)$ has an almost linear behaviour in z for moderate ϵ values).

The solution of (8) for this basic flow is, as suggested by (11) and (13),

$$\hat{\psi} = \frac{n_\epsilon^{\frac{1}{2}}(z)}{\bar{n}_\epsilon^{\frac{1}{2}}}\left[A_1 \exp\left(\frac{Kn_\epsilon^*(z)}{\bar{n}_\epsilon}\right) + A_2 \exp\left(\frac{-Kn_\epsilon^*(z)}{\bar{n}_\epsilon}\right)\right], \tag{15}$$

where $K = 2k$ and

$$n_\epsilon^*(z) = \int_0^z n_\epsilon(z)\,dz = [1-(1-\epsilon z)^{\frac{1}{3}}]/\tfrac{1}{3}\epsilon,$$

so that $n_\epsilon^*(1\cdot0) = \bar{n}_\epsilon$. The standard Eady solution for the flow $u = z$ occurs in the limit $\epsilon \to 0$. It is important to note that ϵ is not necessarily small but lies in the range $-\infty < \epsilon < 1$.

† The negative ϵ value corresponding to a positive ϵ_1 value is given by $-\epsilon_1/(1-\epsilon_1)$.

The eigenvalue problem

Application of the boundary condition, equation (9), to the expression in (15) yields a secular equation for the complex wave speed c. The equation is most conveniently written as

$$A\tilde{c}^2+B\tilde{c}+C = 0, \tag{16}$$

where

$$A = aK^2+(a-1)^2C, \quad B = -K^2-2(a-1)C, \quad C = K\coth K-1,$$

$\tilde{c} = \mathscr{S}_\epsilon c/\mathscr{S}_0\overline{n}_\epsilon$ and $a = (1-\epsilon)^{\frac{1}{2}}$.† The algebraic identity $\overline{n}_\epsilon = (1-a)/\frac{1}{3}\epsilon$ has been used to simplify (16) to this simple algebraic form. The discriminant

$$\delta = (4AC-B^2)^{\frac{1}{2}}$$

of this equation agreeably has the familiar, ϵ-independent, form

$$\delta = 4k[(k-\tanh k)(\coth k-k)]^{\frac{1}{2}}. \tag{17}$$

Amplifying unstable waves occur when the imaginary part of c is positive and non-zero. This occurs when $k < k_N$, the critical neutral short-wave cut-off wavenumber being given by $k_N = \coth k_N$ from (17), i.e. $k_N = 1{\cdot}1997$. That this condition should be independent of ϵ and thus the same as in Eady's theory is due to the particular choice of $\mathscr{N}_\epsilon H/f$ as the length scale for α and to the choice of $\overline{n}_\epsilon$ as the integral scale factor. Thus the dimensional short-wave cut-off is the same as Eady's provided that it is based on the vertically averaged value of the gravitational frequency $N(z)$.

The coefficient A can also be written in the form

$$A = 3C[1+\gamma(K)a\overline{n}_\epsilon]/\overline{n}_\epsilon, \tag{18}$$

where $\gamma(K) = K^2/3C-1$ is a weak dispersion parameter. Then the amplification velocity can be written as

$$c_i = c_{i0}(1+\gamma)/(1+\gamma a\overline{n}_\epsilon), \tag{19}$$

where c_{i0} is the value of c_i in the $\epsilon = 0$ (Eady) case. The wavenumber k_M for the maximum value of the growth rate, $2kc_i$, has the value $k_{M0} = 0{\cdot}8031$ in the $\epsilon = 0$ case. The associated value of $\gamma(K)$ is $\gamma(k_{M0}) = 0{\cdot}1606$. In the $\epsilon \neq 0$ case the value of k_M also depends on ϵ but this dependency is relatively weak, so that to a good approximation $k_M \simeq k_{M0}$ and $\gamma(k_M) \simeq \gamma(k_{M0})$. Use of these values in (19) provides an accurate value for $c_i(k_M)$, as does the further approximated form of (19) $c_i(k_M) \simeq c_{i0}(k_{M0})$. See table 1. This simple result is due to the choice of $\overline{n}_\epsilon^2$ as the scale factor for $\mathscr{S}_\epsilon/\mathscr{S}_0$.

The real part c_r of the phase velocity c is given by

$$c_r = \frac{\mathscr{S}_0}{\mathscr{S}_\epsilon}\frac{\overline{n}_\epsilon}{2a}\left\{1-\frac{\overline{n}_\epsilon(1-a^2)}{3(1+\gamma a\overline{n}_\epsilon)}\right\}, \tag{20}$$

when $k < k_N$. (Note that $\mathscr{S}_0\overline{n}_\epsilon/a\mathscr{S}_\epsilon = u(1{\cdot}0)$). For small values of k, $\gamma \to 0$ in (20) and the resulting expression is algebraically equal to the vertically averaged mean current

$$\overline{u} = \int_0^1 u\,dz.$$

† Although $\overline{n}_\epsilon/\overline{s}_\epsilon$ is equal to a, these relations are purely algebraic.

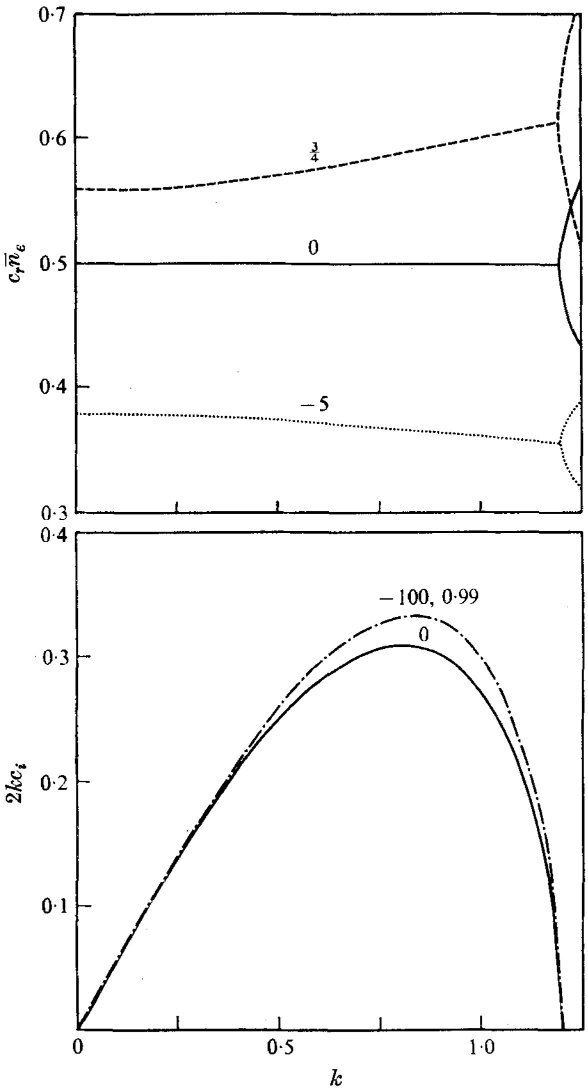

FIGURE 2. Variation with wavenumber k of (a) the normalized wave speed $c_r\bar{n}_\epsilon$ and (b) the normalized growth rate $2kc_i$, for values of ϵ as indicated.

ϵ	z_s	k_M	$c_r(k_M)\mathscr{S}_\epsilon$	$u(z_s)\mathscr{S}_\epsilon$	$\bar{u}\mathscr{S}_\epsilon$	$c_i(k_M)$
0·75	0·593	0·806	0·871	0·866	0·825	0·194
0·50	0·548	0·804	0·677	0·675	0·661	0·193
0·25	0·520	0·804	0·571	0·570	0·565	0·193
0	0·500	0·803	0·500	0·500	0·500	0·193
−0·5	0·472	0·804	0·409	0·409	0·413	0·193
−2·0	0·425	0·806	0·278	0·278	0·285	0·193
−5·0	0·383	0·810	0·179	0·180	0·186	0·194

TABLE 1

Thus $\overline{u}$ (see table 1) is a good approximation to c_r for small k. This result is also suggested by the divergence equation

$$\hat{w}_z = -i\alpha K^2(u-c)\,\hat{\psi}, \tag{21}$$

which when integrated leads to the integral equation for c:

$$c = \overline{u\hat{\psi}}/\overline{\hat{\psi}}. \tag{22}$$

From (22) it is clear that when $c_i \ll c_r$ and variations in $\hat{\psi}$ are small the result $c_r \simeq \overline{u}$ follows.

The approximation $c_r \simeq \overline{u}$ is not particularly accurate when k is not small. To obtain a good approximation when $k = k_M$ requires that the $\hat{\psi}$ variation in (22) be allowed for. When this is done it is found that a good empirical approximation to $c_r(k_M)$ of the form of (22) is given by the weighted mean velocity $\overline{u}_1$, where

$$\overline{u}_1 = \overline{u(1+n_\epsilon^{\frac{1}{2}})}/\overline{1+n_\epsilon^{\frac{1}{2}}}. \tag{23}$$

Another expression which provides a good approximation to $c_r(k_M)$ and which can also be considered as an approximation to (23) is

$$\overline{u}_2 = \overline{u(1+n_\epsilon^{\frac{1}{2}})}/(1+\overline{n}_\epsilon^{\frac{1}{2}}). \tag{24}$$

Both $\overline{u}_1$ and $\overline{u}_2$ seem to be equally good approximations to $c_r(k_M)$. The form $\overline{u}_2$, however, is particularly valuable as it reduces algebraically to the expression

$$\overline{u}_2 = \mathscr{S}_0\mathscr{S}_\epsilon^{-1}[\overline{n}_\epsilon^{\frac{1}{2}}-1]/\tfrac{1}{3}\epsilon. \tag{25}$$

Comparing (14) and (25) suggests that $\overline{u}_2$ is equal to $u(z_s)$, where z_s is the height at which $N(z)$ is equal to its mean value, i.e. where

$$n_\epsilon(z_s) = \overline{n}_\epsilon. \tag{26}$$

Thus $u(z_s)$ provides an accurate approximation to the wave speed $c_r(k_M)$ and so z_s, as defined by (26), is the so-called steering level of the wave.†

Values of $z_s = [1-\overline{n}_\epsilon^{-\frac{1}{2}}]/\epsilon$, k_M, $c_r(k_M)$ and $c_i(k_M)$ together with their approximate forms are listed in table 1. The growth rate and wave speed are shown as functions of k and ϵ in figure 2. It is apparent from figure 2 that c_r is only weakly dependent on k and then only for moderate ϵ values. The growth rate $2kc_i$ is very weakly dependent on ϵ and requires extreme ϵ values to reveal the deviations.

Character of the solution

For unstable waves, the pressure and temperature can be written as

$$\hat{\psi} = \frac{n_\epsilon^{\frac{1}{2}}(z)}{\overline{n}_\epsilon^{\frac{1}{2}}}\left[\frac{Kc}{\overline{n}_\epsilon}\cosh\left(\frac{K}{\overline{n}_\epsilon}n_\epsilon^*(z)\right)-\left(\frac{\mathscr{S}_0}{\mathscr{S}_\epsilon}+\frac{\epsilon}{3}c\right)\sinh\left(\frac{K}{\overline{n}_\epsilon}n_\epsilon^*(z)\right)\right], \tag{27}$$

$$\hat{\psi}_z = \frac{n_\epsilon^{\frac{3}{2}}(z)}{\overline{n}_\epsilon^{\frac{1}{2}}}\left\{\frac{K}{\overline{n}_\epsilon}\left[\frac{Kc}{\overline{n}_\epsilon}\sinh\left(\frac{K}{\overline{n}_\epsilon}n_\epsilon^*(z)\right)-\left(\frac{\mathscr{S}_0}{\mathscr{S}_\epsilon}+\frac{\epsilon}{3}c\right)\cosh\left(\frac{K}{\overline{n}_\epsilon}n_\epsilon^*(z)\right)\right]+\frac{\epsilon}{3}\overline{n}_\epsilon^{\frac{1}{2}}\hat{\psi}\right\}. \tag{28}$$

The most influential modifying factors in this solution are the $n_\epsilon^{\frac{3}{2}}(z)$ and $n_\epsilon^2(z)$ terms that occur in $\hat{\psi}_z$. These factors are strongly variable functions of ϵz and

† For waves with k small and $c_r \simeq \overline{u}$ the steering level is given by $n_\epsilon^{\frac{1}{2}}(z_s) = \overline{n_\epsilon^{\frac{1}{2}}}$.

greatly influence the amplitude of $\hat{\psi}_z$, which thus tends to reflect the variations in the static stability. These factors, however, do not affect the phase of $\hat{\psi}_z$. The phase and amplitude of $\hat{\psi}$ are only weakly influenced by the $n_\epsilon^{\frac{1}{2}}(z)$ factor, so that $\hat{\psi}$ is similar to that of the classical Eady solution.

Figure 3 illustrates the phase and amplitude distributions for various values of ϵ. The amplitudes of all variables are no longer symmetrical about $z = \frac{1}{2}$ when $\epsilon \neq 0$, but tend to have maximum values in $z > \frac{1}{2}$ for $\epsilon > 0$ and in $z < \frac{1}{2}$ for $\epsilon < 0$. The largest effects occur in the amplitudes of $\hat{\psi}_z$ and $\hat{w}_z$, which are considerably enhanced at $z = 1$ ($\epsilon > 0$) and at $z = 0$ ($\epsilon < 0$). Moderate variations with ϵ are evident in the phases of $\hat{w}$ and $\hat{w}_z$ but the phases of $\hat{\psi}$ and $\hat{\psi}_z$ are only weakly dependent on ϵ.

5. Baroclinic annulus waves

The solutions obtained above appear to be particularly relevant to the laboratory experiments on baroclinic instability.† In the annulus the static stability and shear decrease with height and thus have distributions corresponding to $\epsilon < 0$. Horizontal averages, denoted by $\overline{(\ )}^{xy}$, of the finite amplitude three-dimensional u_z and T_z fields obtained in a numerical solution for a baroclinic annulus wave (Williams 1971) are plotted in figure 4. The curves indicate that $\overline{u}_z^{xy}$ and $\overline{T}_z^{xy}$ have almost parallel‡ distributions above the region influenced by the Ekman layer. Thus the generalized Eady approximation Fu_z constant is valid for this flow. Both distributions seem to approach $s_\epsilon(z)$ most satisfactorily when $\epsilon \simeq -4$.

The theoretical solution for $\epsilon = -4$ (similar to the $\epsilon = -5$ one in figure 3) does indeed resemble the numerical solution in the region above the Ekman layer in the central $y = \frac{1}{2}$ plane of the annulus, see figure 9 of Williams (1971). The steering level in this plane occurs at $z_s = 0{\cdot}37$ in the numerical solution and this is close to the value 0·39 for the $\epsilon = -4$ solution. For $\epsilon = -4$ the integral factor $\overline{n}_\epsilon$ is equal to 0·53, so that the dimensional cut-off wavenumber would appear to be about twice as large as that predicted by Eady's theory if it were based on the maximum value of $N(z)$ rather than on the integral mean value of $N(z)$. Such seems to be the situation in the multiple laboratory experiments (P. Mason, personal communication). These results suggest that some annulus waves are a form of generalized Eady wave with a character comparable to the $\epsilon = -4$ solution. Although it is unlikely that the actual annulus waves could correspond identically to these simple functions or that they possess a form equivalent to a unique ϵ, the above identification of the importance of the $n_\epsilon^2(z)$ variations does provide a step towards a more complete theory for annulus convection. An alternative approach is to use the more complex results given in the appendix for whatever actual static-stability–shear distributions are observed or postulated for the system.

† A system where f is independent of y and for whose flows precise data exist.

‡ For the corresponding axisymmetric solution this parallelism is not as strong. This suggests that finite amplitude waves act so as to make $q_y \to 0$ by bringing u_z and dT_s/dz into line.

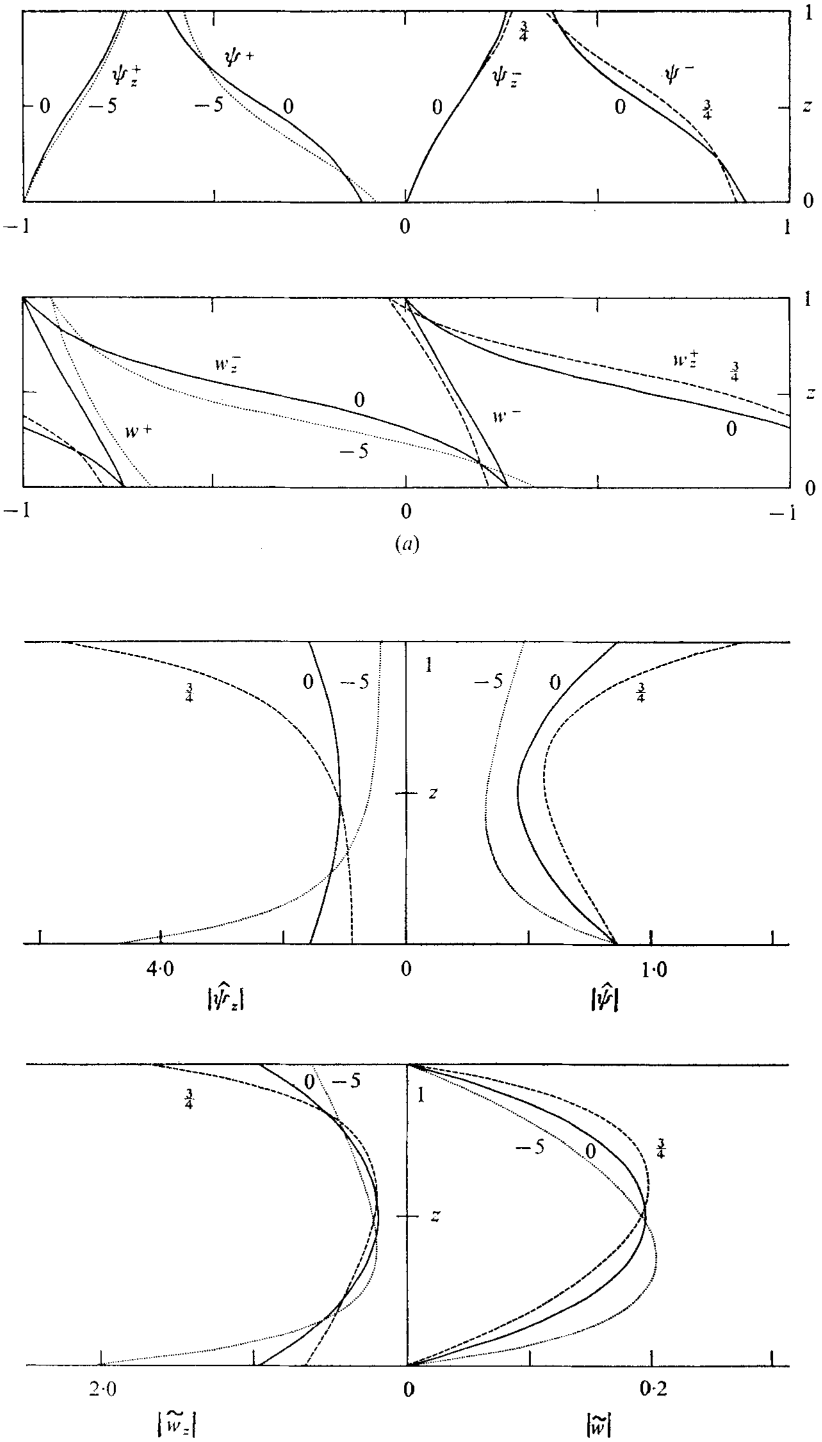

FIGURE 3. (*a*) Phase distributions for the generalized Eady wave, equations (23) and (24), at k_M for representative values of ϵ as indicated and. The reference case $\epsilon = 0$ is the standard Eady wave. Phases are measured relative to that of $\hat{\psi}_z$ at $z = 0$, which is independent of ϵ, in units of π. For clarity, distributions for $\epsilon > 0$ are shown for the right half of the wave and for $\epsilon < 0$ for the left half only. (*b*) Amplitude distributions for the generalized Eady wave at k_M for representative values of ϵ as indicated. The reference case $\epsilon = 0$ is the standard Eady wave. $\tilde{w} = k\,\hat{w}\bar{n}_\epsilon^{\frac{3}{2}}/\alpha$ is the normalized vertical velocity and $\tilde{w}_z = k\hat{w}_z\bar{n}_\epsilon^{\frac{3}{2}}/\alpha$ is the normalized divergence. $\tilde{\psi} = \bar{n}_\epsilon^2\hat{\psi}$ is the normalized stream function and $\tilde{\psi}_z = \bar{n}_\epsilon^2\hat{\psi}$ is the normalized temperature.

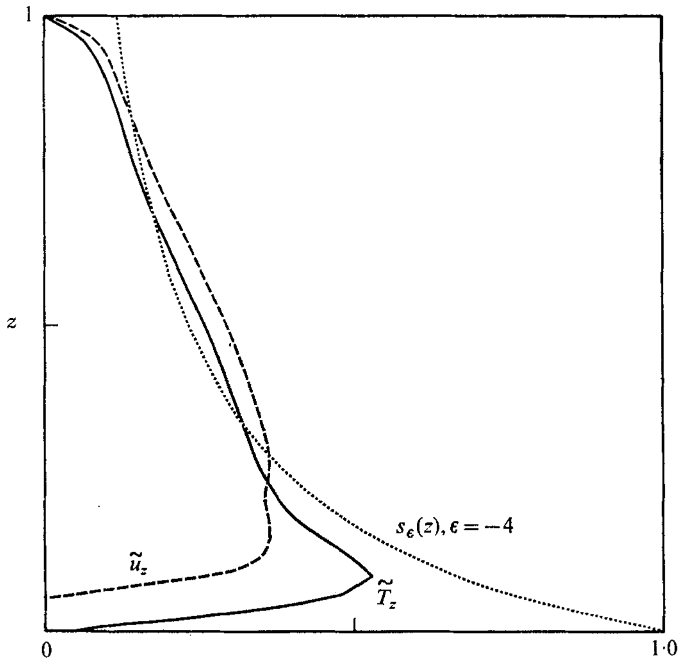

FIGURE 4. Vertical profiles of the horizontally averaged values of the u_z and T_z fields of the numerical annulus wave solution of Williams (1971) and the theoretical curve $s_\epsilon(z)$ for $\epsilon = -4$. The annulus profiles are arbitrarily scaled and are of dimensional quantities $\tilde{u}_z = 3\bar{u}_z^{xv}$ and $\tilde{T}_z = \bar{T}_z^{xv}/\Delta T$ of the numerical solution.

A complete theory for annulus waves requires a synthesis of the effects of the major processes active in the system. Clearly the predominant mechanisms are the baroclinic instability mechanism of the type discussed above and the boundary-layer dynamics of the basic state. The important modifying processes include the Ekman-layer dissipation of the wave and the nonlinearities of the finite amplitude wave.

6. Conclusion

Eady-type solutions have been found for flows with height-varying static stability and shear. The solutions are exact when this variation is of the form $(1-\epsilon z)^{-\frac{1}{2}}$, with the classical Eady solution corresponding to the case $\epsilon = 0$. Comparison between the theory and a numerical solution for annulus waves indicates that such variations are physically significant and that annulus waves are generalized Eady waves comparable to the $\epsilon = -4$ solutions. This conclusion is supported by the asymptotic value of the instability criterion obtained in the laboratory experiments. This value is comparable to the $\epsilon = -4$ theoretical value when it is based on the vertically averaged gravitational frequency.

I should like to thank Dr Brian Hoskins for his valuable comments on this work.

Appendix. General WKB solutions

When the static stability and shear do not have the simple functional form discussed in § 4, the general WKB solution (11) can be used to solve the eigenvalue problem. The results, given below, lack the algebraic relations and reducibility of the generalized exact solutions and are therefore more complex and their affinity to the Eady solution less obvious. However, the solutions have a physical and practical value, being easy to evaluate numerically.

The general solution for $\hat{\psi}$ can be written as

$$\hat{\psi} = \frac{n_\epsilon^{\frac{1}{2}}(z)}{\overline{n}_\epsilon^{\frac{1}{2}}} \left\{ \left[\frac{K}{\overline{n}_\epsilon} n_\epsilon^2(0)\,[c-u(0)]\right] \cosh\left(\frac{K}{\overline{n}_\epsilon} n_\epsilon^*(z)\right) - \{\mathscr{S}_0 \mathscr{S}_\epsilon^{-1} s_\epsilon(0)\, n_\epsilon(0) + \tfrac{1}{2} n_\epsilon'(0)\,[c-u(0)]\} \sinh\left(\frac{K}{\overline{n}_\epsilon} n_\epsilon^*(z)\right)\right\}, \quad (\mathrm{A}\,1)$$

when it is based on the values of the basic functions at $z = 0$ and where the gradient function $n_\epsilon'(z) = dn_\epsilon(z)/dz$ has been introduced.

The secular equation $Ac^2 + Bc + C = 0$ has the coefficients

$$A = \frac{K^{2\dagger}}{\overline{n}_\epsilon^2} + \tfrac{1}{2}K(\coth K)\,\Delta\left(\frac{n_\epsilon'}{n_\epsilon^2}\right) - \frac{1}{4}\overline{\left(\frac{n_\epsilon'}{n_\epsilon^2}\right)^{2}}^{g}, \quad (\mathrm{A}\,2)$$

$$B = -2\overline{u}^a A^\dagger + \frac{\mathscr{S}_0}{\mathscr{S}_\epsilon}\frac{K}{\overline{n}_\epsilon}(\coth K)\,\Delta\left(\frac{s_\epsilon}{n_\epsilon}\right) - \frac{\mathscr{S}_0}{\mathscr{S}_\epsilon}\overline{\left(\frac{s_\epsilon}{n_\epsilon}\right)^{2}}^{g}\,\overline{\left(\frac{n_\epsilon'}{s_\epsilon n_\epsilon}\right)}^{a}, \quad (\mathrm{A}\,3)$$

$$C = \overline{u^2}^g A^\dagger + \frac{\mathscr{S}_0^\dagger K}{\mathscr{S}_\epsilon \overline{n}_\epsilon} \coth K\, \overline{\left(\frac{s_\epsilon}{n_\epsilon}\right)^{2}}^{g} \Delta\left(\frac{u n_\epsilon}{s_\epsilon}\right) - \frac{\mathscr{S}_0}{\mathscr{S}_\epsilon}\overline{\left(\frac{s_\epsilon}{n_\epsilon}\right)^{2}}^{g}\left\{\frac{\mathscr{S}_0}{\mathscr{S}_\epsilon} - \overline{\left(\frac{u n_\epsilon'}{s_\epsilon n_\epsilon}\right)}^{a}\right\}, \quad (\mathrm{A}\,4)$$

where the arithmetic and geometric means of values at the two boundaries have been written as, for example, $\overline{u}^a = \frac{1}{2}\{u(1)+u(0)\}$ and $\overline{u}^g = \{u(0)\,u(1)\}^{\frac{1}{2}}$ respectively and the difference as, for example, $\Delta u = u(1) - u(0)$. With this notation the real and imaginary parts of the wave velocity c can be written as

$$c_r = \overline{u}^{a\dagger} - \frac{\mathscr{S}_0}{2A}\left[\frac{K}{\overline{n}_\epsilon}(\coth K)\,\Delta\left(\frac{s_\epsilon}{n_\epsilon}\right) - \overline{\left(\frac{s_\epsilon}{n_\epsilon}\right)^{2}}^{g}\,\overline{\left(\frac{n_\epsilon'}{s_\epsilon n_\epsilon}\right)}^{a}\right] \quad (\mathrm{A}\,5)$$

and

$$c_i^2 = -\frac{\mathscr{S}_0^2}{4}\overline{s}_\epsilon^{2\dagger} + \frac{\mathscr{S}_0^2}{A}\overline{s}_\epsilon\left[\frac{K^\dagger}{\overline{n}_\epsilon}\coth K\,\overline{\left(\frac{s_\epsilon}{n_\epsilon}\right)}^{a} + \frac{1}{4}\overline{\left(\frac{s_\epsilon}{n_\epsilon}\right)^{2}}^{g}\Delta\left(\frac{n_\epsilon'}{s_\epsilon n_\epsilon}\right)\right] - \frac{\mathscr{S}_0^2}{4A^2}\left[\frac{4K^2}{\overline{n}_\epsilon^2}\overline{\left(\frac{s_\epsilon}{n_\epsilon}\right)^{2}}^{g\dagger} + \left\{\frac{K}{\overline{n}_\epsilon}(\coth K)\,\Delta\left(\frac{s_\epsilon}{n_\epsilon}\right)\right\}^2 + \frac{1}{4}\left\{\overline{\left(\frac{s_\epsilon}{n_\epsilon}\right)^{2}}^{g}\Delta\left(\frac{n_\epsilon'}{s_\epsilon n_\epsilon}\right)\right\}^2 + \frac{2K}{\overline{n}_\epsilon}\coth K\,\overline{\left(\frac{s_\epsilon}{n_\epsilon}\right)^{2}}^{g}\,\overline{\left(\frac{s_\epsilon}{n_\epsilon}\right)}^{a}\Delta\left(\frac{n_\epsilon'}{s_\epsilon n_\epsilon}\right)\right]. \quad (\mathrm{A}\,6)$$

Some of the quantities occurring in (A 2)–(A 6) can be interpreted as being boundary values of local parameters, e.g. $n_\epsilon^2(z)/s_\epsilon^2(z)$ is a local relative Richardson number and an inverse measure of the basic available potential energy; both of these quantities are independent of z in the standard Eady problem. The identity $n_\epsilon^2(z) = s_\epsilon(z)$ can be used in the above equations to provide some algebraic reduction.

The terms in (A 2)–(A 6) that give the Eady solution have been marked with a dagger. Various approximations or specializations to these equations are possible, including the reduction to the exact case of § 4.

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